Synchronization of Innovation Cycles

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Introduction

- Many models of nonlinear dynamics generating endogenous fluctuations (in innovation), but they are all (effectively) one-country, one-sector models

- Need for **multi-sector extensions** (with endogenous fluctuations in each sector)
  To evaluate the aggregate effects, we need to know how fluctuations at different sectors affect each other.
  ✓How do sectors co-move? Are they ***synchronized to amplify*** fluctuations? ***Or asynchronized to moderate***?
  ✓No previous work on these issues, neither theoretically nor empirically.
  ✓We need a conceptual framework to guide our theoretical and empirical research.

- Need for **multi-country extensions** (with endogenous fluctuations in each country)
  Theoretical Motivation: Most macroeconomists study the effects of globalization in a model where productivity movements are driven by some exogenous processes. But, globalization can change
  ✓Productivity growth rate, as already shown by endogenous growth models.
  ✓Synchronicity of productivity fluctuations, in a model of endogenous cycles
  Empirical Motivation: More bilateral trade leads to more synchronized business cycles among developed countries, but *not* between developed & developing countries.
  ✓Hard to explain this "***trade-comovement puzzle***" in models with exogenous shocks
  ✓Easier to explain in models with endogenous sources of fluctuations

- **Our building block**: Deneckere-Judd (1992) one-sector, closed economy model of endogenous innovation fluctuations, characterized by a **skew-tent map**
  - ✓Mathematically, our extensions generate **coupled skew-tent maps**.
  - ✓Conceptually, this is a study of **weakly coupled oscillators**.

Synchronization of Weakly Coupled Oscillators

Natural Science: A Major Topic. Thousands of examples: Just to name a few,

- Christiaan Huygens' pendulum clocks
- The Moon's rotation and revolution
- London Millennium Bridge

Also, search videos "Synchronization of Metronomes" on Youtube!

But, we cannot use existing models in the natural science. They simply append an additional term that captures "synchronizing forces," multiplied by "a coupling parameter," and study the effects of changing a coupling parameter.   Without micro foundations,

- no structural interpretation can be given to the "coupling parameter."
- subject to the "Lucas critique". In general equilibrium, such coupling would change innovation incentives.

The Building Block: Deneckere-Judd (1992)

Judd (1985); Dynamic Dixit-Stiglitz monopolistic competitive model;
Innovators pay fixed cost to introduce a new (horizontally differentiated) variety

Judd (1985; Sec.2); They earn the monopoly profit forever. Converging to steady state

Main Question: What if innovators have monopoly only for a limited time?

- Each variety sold initially at monopoly price; later at competitive price
- Impact of an innovation, initially muted, reach its full potential *with a delay*
- Past innovation discourages innovators more than contemporaneous innovation
- **Temporal clustering of innovation**, leading to aggregate fluctuations

Judd (1985; Sec.3); *Continuous time* and monopoly lasting for $0 < T < \infty$

- *Delayed differential equation* (with an infinite dimensional state space)
- For $T > T_c > 0$, the economy alternates between the phases of active innovation and of no innovation along any equilibrium path for almost all initial conditions.

Judd (1985; Sec.4); also Deneckere & Judd (1992; DJ for short)

- *Discrete time* and *one period monopoly* for analytical tractability
- **1D state space** (the measure of competitive varieties inherited from past innovation determines how saturated the market is)
- Unique equilibrium path generated by **1D PWL noninvertible (i.e., skew-tent) map**.
- When the unique steady state is unstable, fluctuations for almost all initial conditions, converging either to a ***2-cycle*** or to a *chaotic attractor*

Revisiting Deneckere-Judd (1992)

Time: $t \in \{0,1,2,...\}$

Final (Consumption) Good Sector: assembles differentiated inputs a la Dixit-Stiglitz

$$Y_t = X_t = \left[\int_{\Omega_t} [x_t(v)]^{1-\frac{1}{\sigma}} dv \right]^{\frac{\sigma}{\sigma-1}} \qquad (\sigma > 1)$$

Demand for Differentiated Inputs:

$$\frac{x_t(v)}{X_t} = \left(\frac{p_t(v)}{P_t} \right)^{-\sigma}, \qquad \text{where } [P_t]^{1-\sigma} = \int_{\Omega_t} [p_t(v)]^{1-\sigma} dv$$

Set of Differentiated Inputs: can change due to *Innovation* and *Obsolescence*.

$\Omega_t = \Omega_t^c + \Omega_t^m$; Set of all differentiated inputs available in t
Ω_t^c: Set of competitively supplied inputs inherited in period t.
Ω_t^m: Set of new inputs introduced and sold exclusively by their innovators for one period.

Differentiated Inputs Pricing: ψ units of labor (the numeraire) for producing one unit of each variety:

$$p_t(\nu) = \psi \equiv p_t^c; \qquad x_t(\nu) \equiv x_t^c \qquad \text{for } \nu \in \Omega_t^c \subset \Omega_t$$

$$p_t(\nu) = \frac{\psi}{1-1/\sigma} \equiv p_t^m; \qquad x_t(\nu) \equiv x_t^m \qquad \text{for } \nu \in \Omega_t^m = \Omega_t - \Omega_t^c$$

$$\frac{p_t^c}{p_t^m} = 1 - \frac{1}{\sigma} < 1; \qquad \frac{x_t^c}{x_t^m} = \left(1 - \frac{1}{\sigma}\right)^{-\sigma} > 1; \quad \frac{p_t^c}{p_t^m}\frac{x_t^c}{x_t^m} = \left(1 - \frac{1}{\sigma}\right)^{1-\sigma} \equiv \theta \in (1, e)$$

Aggregate Output and Price: Let N_t^c (N_t^m) be the measure of Ω_t^c (Ω_t^m)

$$(Y_t)^{1-1/\sigma} = \left[N_t^c (x_t^c)^{1-1/\sigma} + N_t^m (x_t^m)^{1-1/\sigma}\right] = M_t (x_t^c)^{1-1/\sigma},$$

$$(P_t)^{1-\sigma} = \left[N_t^c (p_t^c)^{1-\sigma} + N_t^m (p_t^m)^{1-\sigma}\right] = M_t (\psi)^{1-\sigma}, \qquad \text{where } M_t \equiv N_t^c + \frac{N_t^m}{\theta}.$$

- One competitive variety has the same effect with $\theta > 1$ monopolistic varieties.
- θ is increasing in σ, but varies little for a wide range of σ.

σ	$\to 1$	2	4	5	6	8	10	14	20	$\to \infty$
θ	$\to 1$	2	2.37	2.44	2.49	2.55	2.58	2.62	2.65	$\to e = 2.71828\ldots$

Introduction of New Varieties: Innovation cost per variety, f

$$\pi_t^m \equiv p_t^m x_t^m - \left(\psi x_t^m + f\right) \leq 0; \;\; N_t^m \geq 0 \quad \Rightarrow \quad \psi x_t^c \leq \sigma\theta f; \;\; M_t \geq N_t^c$$

Complementary Slackness: Either net profit or innovation has to be zero in equilibrium

Resource Constraint: Fixed total labor supply, L,

$$L = N_t^c(\psi x_t^c) + N_t^m(\psi x_t^m + f) = N_t^c(p_t^c x_t^c) + N_t^m(p_t^m x_t^m) = \left(\psi x_t^c\right)M_t$$

$$\Rightarrow M_t = \max\left\{\frac{L}{\sigma\theta f}, N_t^c\right\}.$$

Obsolescence of Old Varieties: $\delta \in (0,1)$, **the Survival Rate**

$$N_{t+1}^c = \delta\left(N_t^c + N_t^m\right) = \delta\left(\theta M_t + (1-\theta)N_t^c\right) = \delta \max\left\{\frac{L}{\sigma f} + (1-\theta)N_t^c, N_t^c\right\}$$

Alternatively, labor supply may grow at a constant factor, $G = 1/\delta > 1$.

Let $n_t \equiv (\sigma\theta f)\dfrac{N_t^c}{L}$: (Normalized measure of) competitive varieties per labor supply

Skew-Tent Map

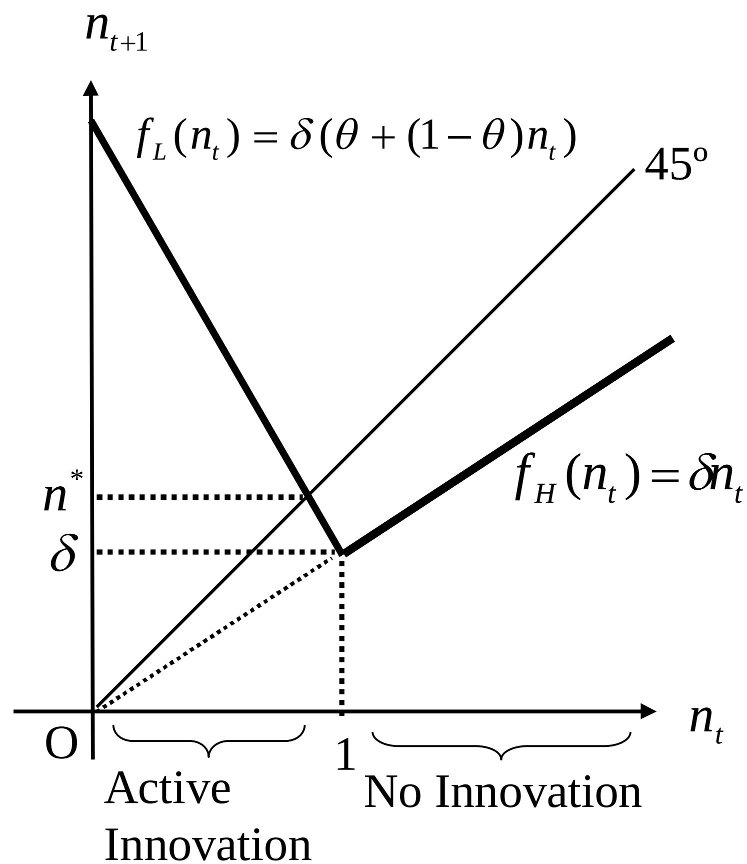
$$n_{t+1} = f(n_t) \equiv \begin{cases} f_L(n_t) \equiv \delta\big(\theta + (1-\theta)n_t\big) & \text{if } n_t < 1 \\ f_H(n_t) \equiv \quad \delta n_t & \text{if } n_t > 1 \end{cases}$$

$\delta \in (0,1)$, Survival rate of each variety due to *obsolescence* (or *exogenous labor supply growth*)

$\theta \equiv \left(1 - \dfrac{1}{\sigma}\right)^{1-\sigma} \in (1,e)$, increasing in σ (EoS)

Market share of a competitive variety relative to a monopolistic variety

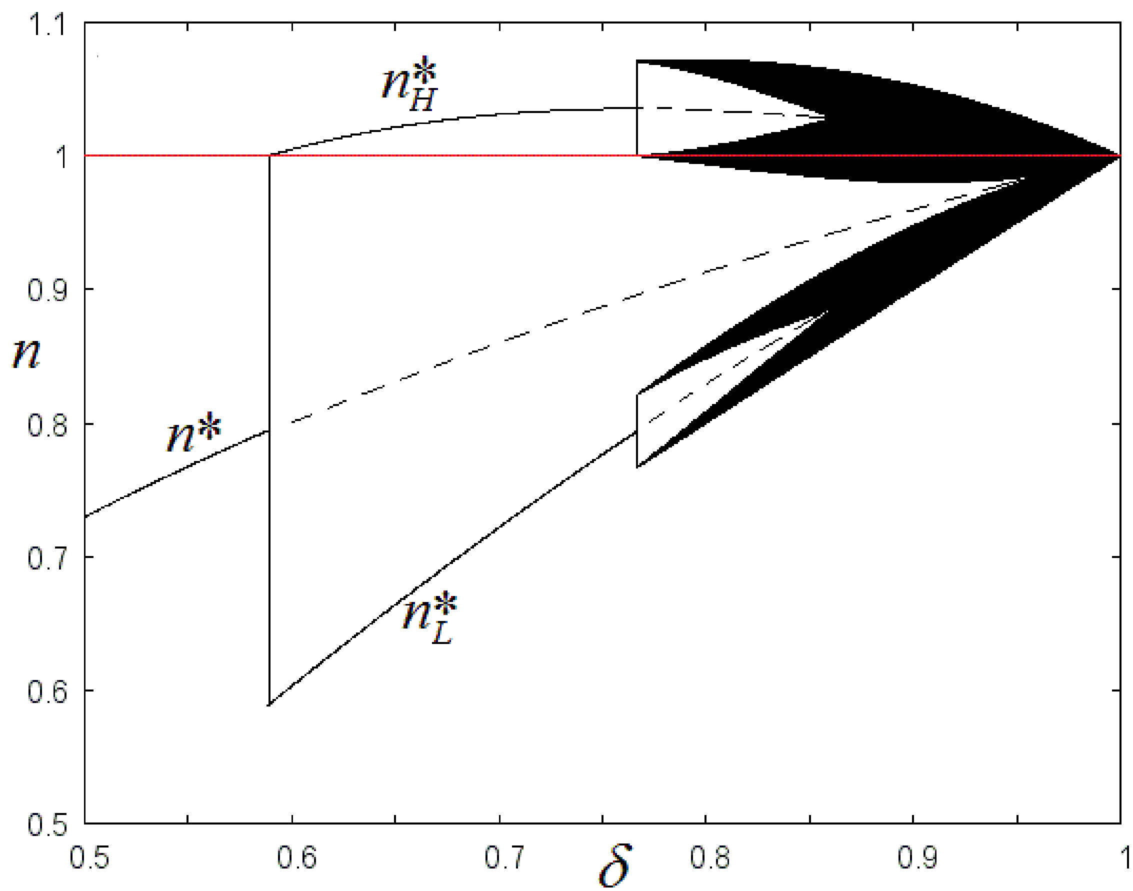
$\theta - 1 > 0$: the delayed impact of innovations



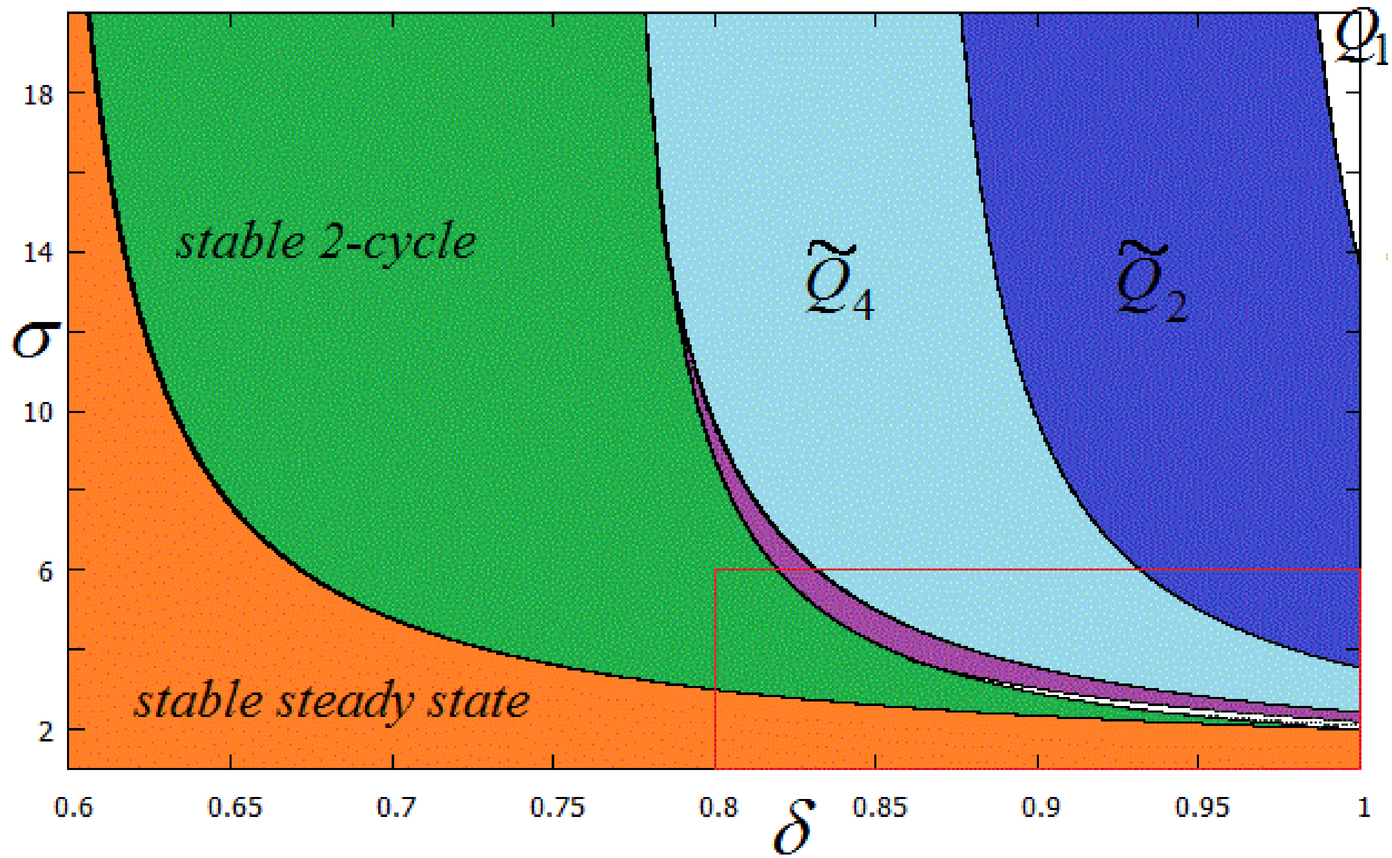
A *Unique* Attractor:

- Stable steady state for $\delta(\theta-1)<1$
- Stable 2-cycle for $\delta^2(\theta-1)<1<\delta(\theta-1)$
- Robust chaotic attractor with 2^m intervals (m = 0, 1,…) for $\delta^2(\theta-1)>1$

Effects of a higher *δ*



In the (σ, δ)-plane Endogenous fluctuations with
- a higher σ (more substitutable; stronger incentive to avoid competition)
- a higher δ (more past innovation survives to crowd out current innovation).



We focus on **the stable 2-cycle case**, $\delta^2(\theta-1)<1<\delta(\theta-1)$.

A Two-Sector Model of Endogenous Innovation Cycles

Based on our “Interdependent Innovation Cycles”

Two-Sector Extension:

- Each sector produces a Dixit-Stiglitz composite, as in Deneckere-Judd
- CES over the two composites with ε (EoS across sectors) < σ (EoS within each sector)

Results

- **2D state space** (the measures of competitive goods in the two sectors determine the current state of the economy)
- Unique equilibrium path generated by **2D-PWS, noninvertible map**
- Dynamics in the two sectors are **decoupled** for Cobb-Douglas ($\varepsilon = 1$). Whether dynamics may converge to either synchronized or asynchronized 2-cycles depends on how you draw the initial condition
- *As ε goes up from one,* fluctuations become **synchronized**
  - *Basin of attraction* for synchronized 2-cycles ***expands and covers the state space.***
  - *Basin of attraction* for asynchronized 2-cycles ***shrinks & disappears***

This occurs before ε reaches *σ.*

- *As ε goes down from one,* fluctuations become **asynchronized**
  - *Basin of attraction* for synchronized 2-cycles ***shrinks***
  - *Basin of attraction* for asynchronized 2-cycles ***expands***

Thus, perhaps surprisingly and counter-intuitively,

- $\varepsilon > 1 \rightarrow$ *synchronization & amplification*
- $\varepsilon < 1 \rightarrow$ *asynchronization & moderation*

2-Dim Dynamical System; $n_{t+1} = F(n_t)$ with $n_t \equiv (n_{1t}, n_{2t}) \in R_+^2$

$$n_{1t+1} = \delta\left(\theta/2 + (1-\theta)n_{1t}\right) \quad \text{for } n_t \in D_{LL} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_j < 1/2\right\}$$
$$n_{2t+1} = \delta\left(\theta/2 + (1-\theta)n_{2t}\right)$$

$$n_{1t+1} = \delta n_{1t} \quad \text{for } n_t \in D_{HH} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_j > g(n_i)\right\}$$
$$n_{2t+1} = \delta n_{2t}$$

$$n_{1t+1} = \delta n_{1t} \quad \text{for } n_t \in D_{HL} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 > 1/2; n_2 < g(n_1)\right\}$$
$$n_{2t+1} = \delta\left(\theta g(n_{1t}) + (1-\theta)n_{2t}\right)$$

$$n_{1t+1} = \delta\left(\theta g(n_{2t}) + (1-\theta)n_{1t}\right) \quad \text{for } n_t \in D_{LH} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 < g(n_2); n_2 > 1/2\right\}$$
$$n_{2t+1} = \delta n_{2t}$$

where

$m_j \equiv g(m_i) < 1$ is defined by $(m_j)^{\eta-1} - (m_j)^{\eta} = (m_i)^{\eta}$ with $\eta \equiv \dfrac{\varepsilon - 1}{\sigma - 1} \in (-\infty, 1)$.

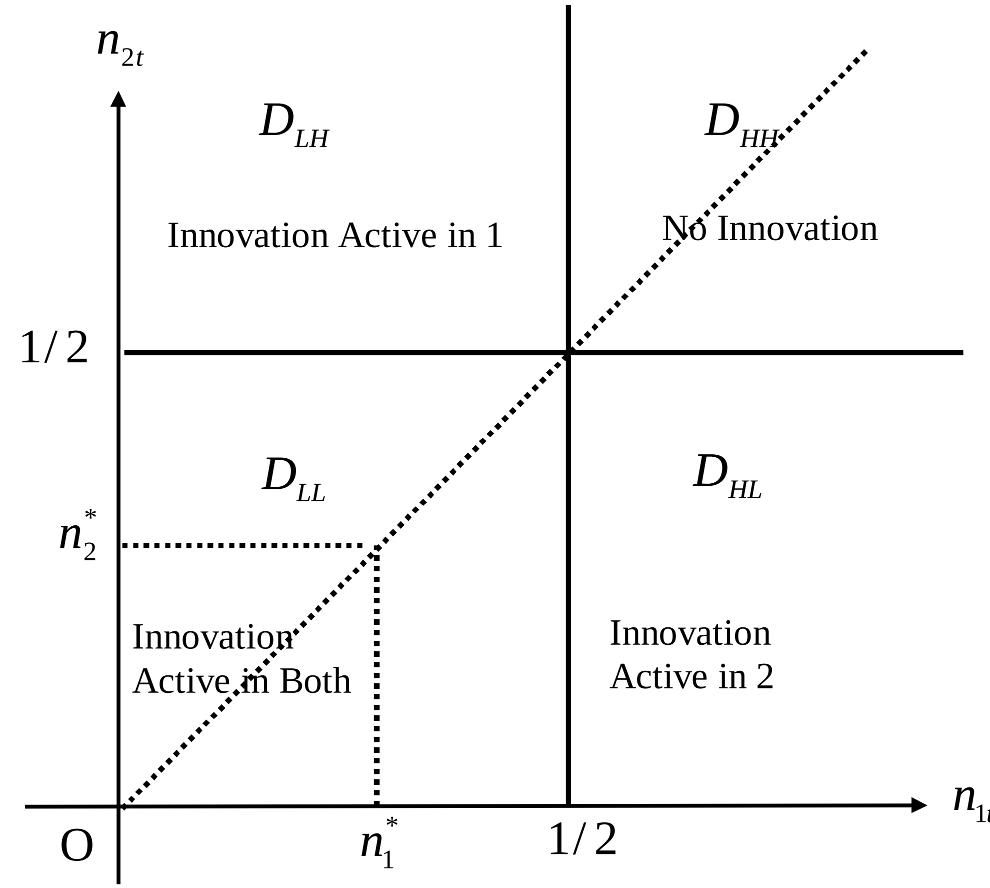
Independent (Decoupled) Skew Tent Maps: Cobb-Douglas Case ($\varepsilon = 1$)

If $\varepsilon = 1$, $\eta = 0$ and $g(m_i) = 1/2$.

2D system consists of two independent 1D skew-tent maps:

$$n_{jt+1} = \delta\left(\theta \max\{1/2, n_{jt}\} + (1-\theta) n_{jt}\right).$$

Unique S.S.: $n_j^* = \dfrac{\theta\delta/2}{1+\delta(\theta-1)} < 1/2$



A *Unique* Attractor:

- Stable steady state for $\delta(\theta-1) < 1$
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- Robust chaotic attractor with 2^m intervals (m = 0, 1, 2,…) for $\delta^2(\theta-1) > 1$

Independent Stable 2-Cycles: $\varepsilon = 1$ $(\eta = 0)$; $\delta(\theta-1) > 1 > \delta^2(\theta-1)$

Each component 1D-map has

- an *unstable* **steady state,** $n_j^* = \dfrac{\theta\delta/2}{1+\delta(\theta-1)} < 1/2$
- a *stable* **2-cycle,** $n_{jL}^* = \dfrac{\delta^2\theta/2}{1+\delta^2(\theta-1)} \leftrightarrow n_{jH}^* = \dfrac{\delta\theta/2}{1+\delta^2(\theta-1)}$

As a 2D-map, this system has

- **An unstable steady state**; (n_1^*, n_2^*).
- **A pair of stable 2-cycles:**
  - **Synchronized;** $(n_{1L}^*, n_{2L}^*) \leftrightarrow (n_{1H}^*, n_{2H}^*)$, with **Basin of Attraction** in Red.
  - **Asynchronized;** $(n_{1L}^*, n_{2H}^*) \leftrightarrow (n_{1H}^*, n_{2L}^*)$, with **Basin of Attraction** in White.
- **A pair of saddle 2-cycles:**

$(n_{1L}^*, n_2^*) \leftrightarrow (n_{1H}^*, n_2^*)$ & $(n_1^*, n_{2H}^*) \leftrightarrow (n_1^*, n_{2L}^*)$.

The closure of the stable sets of the two saddles forms the boundaries of the basins of attraction of the two stable 2-cycles.

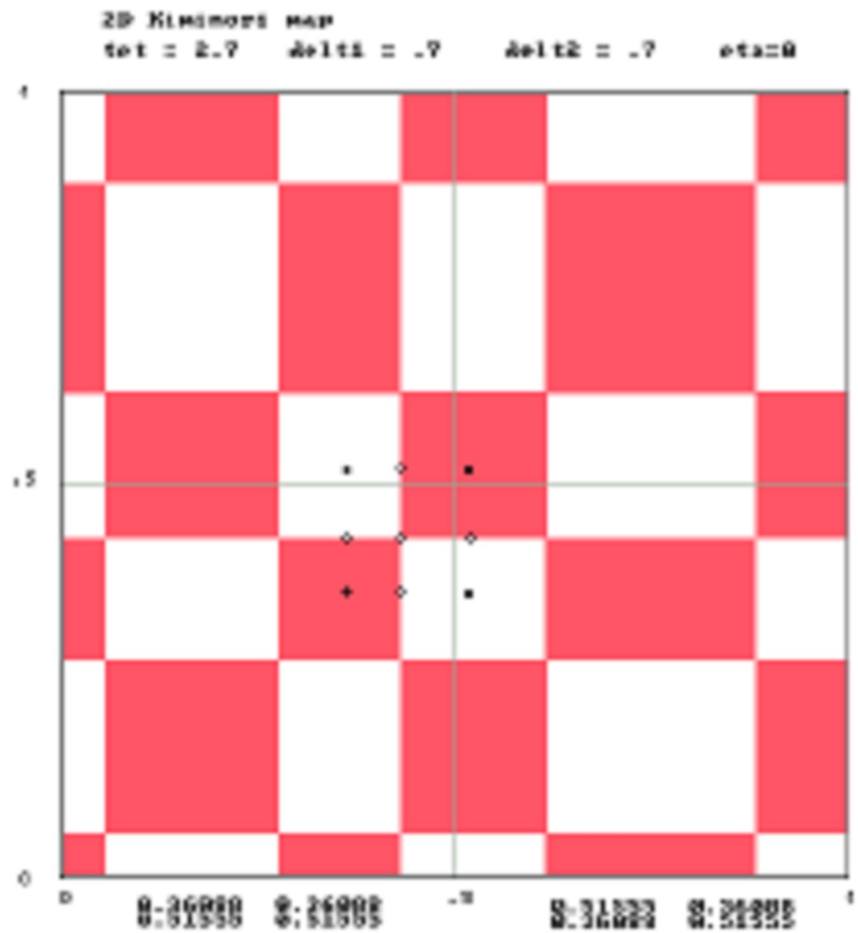
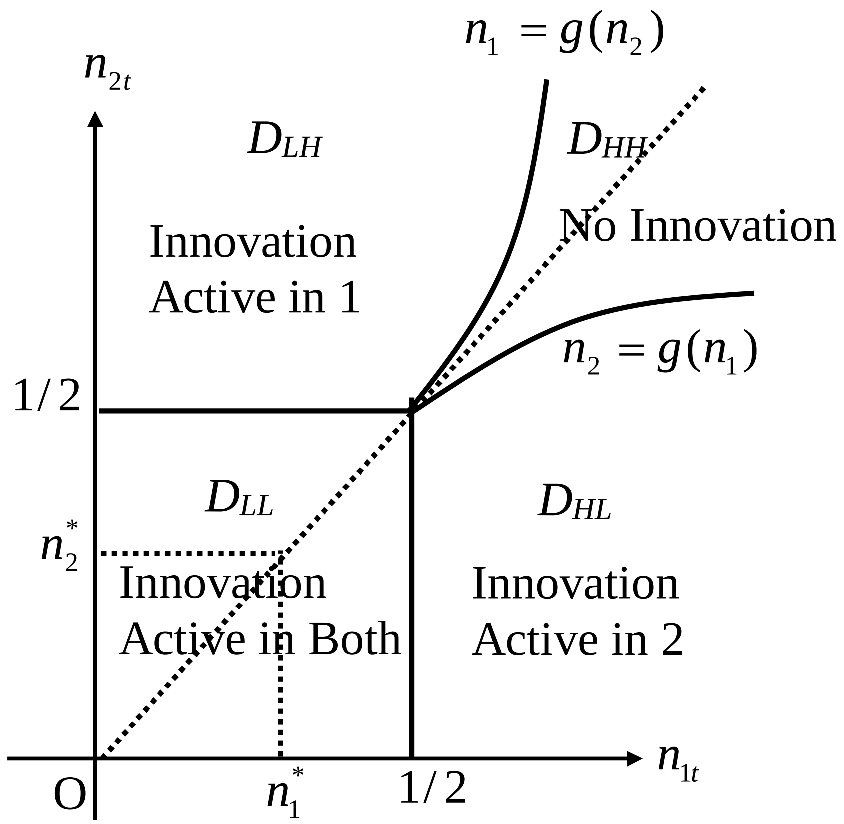


Fig.2a $\delta = 0.7$

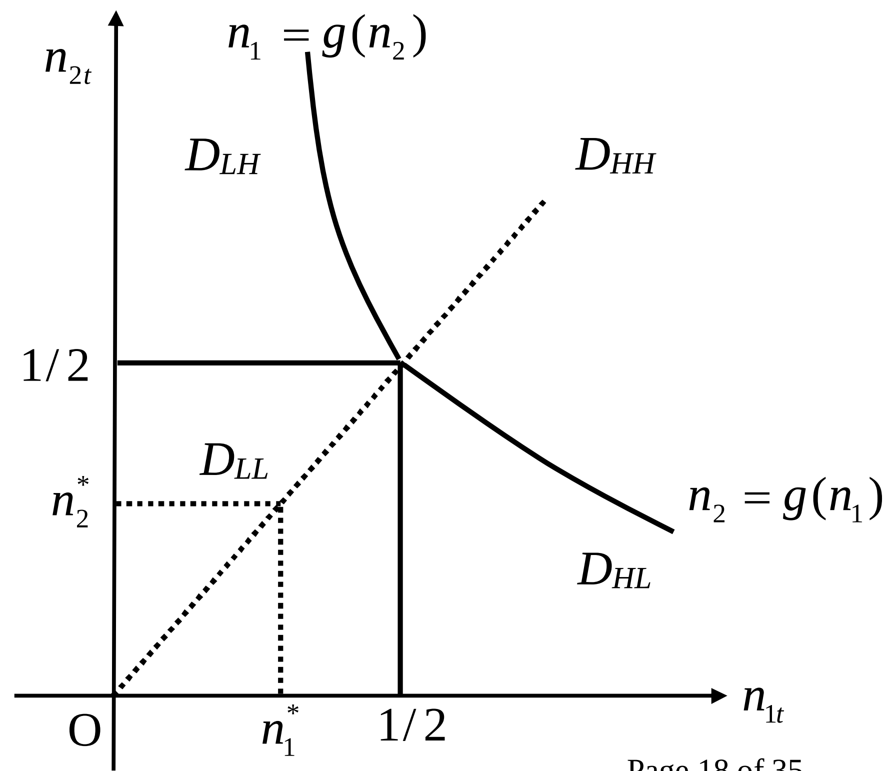
Interdependent (Coupled) Skew Tent Maps: $\varepsilon \neq 1$ $(\eta \neq 0)$

- 2 components still independent in D_{LL} and D_{HH}, which includes the diagonal.
- For $\varepsilon < 1 (\eta < 0)$: $g(\bullet)$ is increasing. More competitive goods in sector 2 (1) increase the market size for sector 1 (2), encourage innovation in 1 in D_{LH} (D_{HL}).
- For $1 < \varepsilon < \sigma$ $(0 < \eta < 1)$: $g(\bullet)$ is decreasing. More competitive goods in sector 2 (1) decreases the market size for sector 2 (1), discourage innovation in 1 in D_{LH} (D_{HL}).

Complements: $\varepsilon < 1$ $(\eta < 0)$



Substitutes: $1 < \varepsilon < \sigma$ $(0 < \eta < 1)$



Interdependent 2-Cycles: $\varepsilon \neq 1$ $(\eta \neq 0)$, with $\delta(\theta-1) > 1 > \delta^2(\theta-1)$

Each component 1D-map has:

- **an *unstable* steady state,** $n_j^* = n^* \equiv \dfrac{\theta\delta/2}{1+(\theta-1)\delta}$
- **a *stable* 2-cycle,** $n_{jL}^* = n_L^* \equiv \dfrac{\delta^2\theta/2}{1+(\theta-1)\delta^2} \leftrightarrow n_{jH}^* = n_H^* \equiv \dfrac{\delta\theta/2}{1+(\theta-1)\delta^2}$.

As a 2D-map,

- As ε (or η) increases, D_{LH} & D_{LH} shrink and D_{HH} expands.
- As ε (or η) decreases, D_{LH} & D_{LH} expand and D_{HH} shrinks.
- **Synchronized 2-cycle,** $(n_L^*, n_L^*) \leftrightarrow (n_H^*, n_H^*)$ exists and stable; not affected by ε (or η).
- **Symmetric Asynchronized 2-cycle,** $(n_L^a, n_H^a) \in D_{LH} \leftrightarrow (n_H^a, n_L^a) \in D_{HL}$, depends on ε (or η), and no longer equal to $(n_L^*, n_H^*) \leftrightarrow (n_H^*, n_L^*)$. It exists for all ε (or η); stable for $\varepsilon < \varepsilon_c$ $(\eta < \eta_c)$ and **unstable** for $1 < \varepsilon_c < \varepsilon < \sigma$ $(0 < \eta_c < \eta < 1)$.

Furthermore, one could see numerically,

- For $\eta < \eta_c$, a higher η expands the basin of attraction for the synchronized 2-cycle.

Symmetric Asynchronized 2-Cycle: $\left(n_L^a, n_H^a\right) \in D_{LH} \leftrightarrow \left(n_H^a, n_L^a\right) \in D_{HL}$

$$n_H^a = \frac{1}{\beta + \beta^{1-\eta}} > \frac{1}{2}; \qquad n_L^a = \frac{\delta}{\beta + \beta^{1-\eta}} = \delta n_H^a < \beta n_H^a = g(n_H^a),$$

where $\beta \equiv \dfrac{1+\delta^2(\theta-1)}{\delta\theta} \in (\delta,1)$ and $m_j \equiv g(m_k)$ solves $(m_j)^{\eta-1} - (m_j)^{\eta} = (m_k)^{\eta}$.

Jacobian at this 2-cycle:

$$J = \delta^2 \begin{bmatrix} 1-\theta+\theta^2\gamma^2 & (1-\theta)\theta\gamma \\ \theta\gamma & 1-\theta \end{bmatrix},$$

where $\gamma \equiv g'(n_H^a) = \beta\left(1 - \dfrac{1}{\eta}(1+\beta^{\eta})\right)^{-1} \equiv \gamma(\eta)$.

Two Eigenvalues:

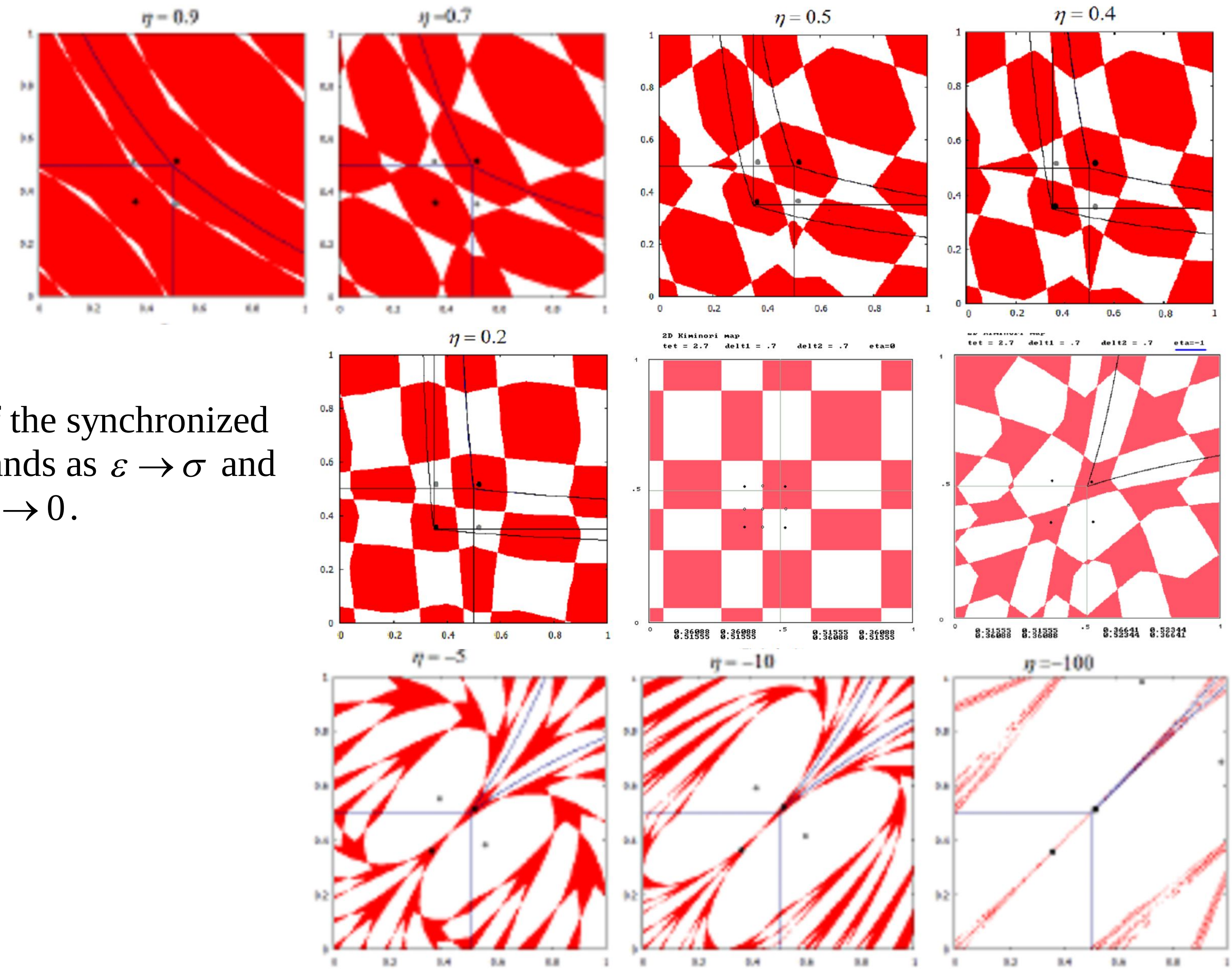
- Complex conjugated if $(\gamma(\eta))^2 < 4(\theta-1)/\theta^2$; **a stable focus,** as $Det(J) = \delta^4(1-\theta)^2 < 1$
- Real, both positive, less than one if $4(\theta-1)/\theta^2 < (\gamma(\eta))^2 < \beta^2$; **a stable node;**
- Real, both positive, one greater than one if $\beta^2 < (\gamma(\eta))^2 < 1$; **an unstable saddle.**

$(\gamma(\eta))^2 < \beta^2$ for $\eta < 0$; increasing in $\eta \in (0,1)$ with $(\gamma(0))^2 = 0$ and $(\gamma(1))^2 = 1$. Hence,

$\exists \eta_c \in (0,1)$, s.t. this 2-cycle is stable for $\eta < \eta_c$ and unstable for $\eta_c < \eta < 1$.

Basins of 2-cycles: Synchronzied (Red) vs. Asynchronized (White)

The basin of the synchronized 2-cycle expands as $\varepsilon \to \sigma$ and shrinks as $\varepsilon \to 0$.



A Two-Country Model of Endogenous Innovation Cycles

Based on our "Globalization and Synchronization of Innovation Cycles"

Helpman & Krugman (1985; Ch.10):

Trade in horizontally differentiated (Dixit-Stiglitz) goods with *iceberg trade costs* between two *structurally identical* countries; only their sizes may be different.

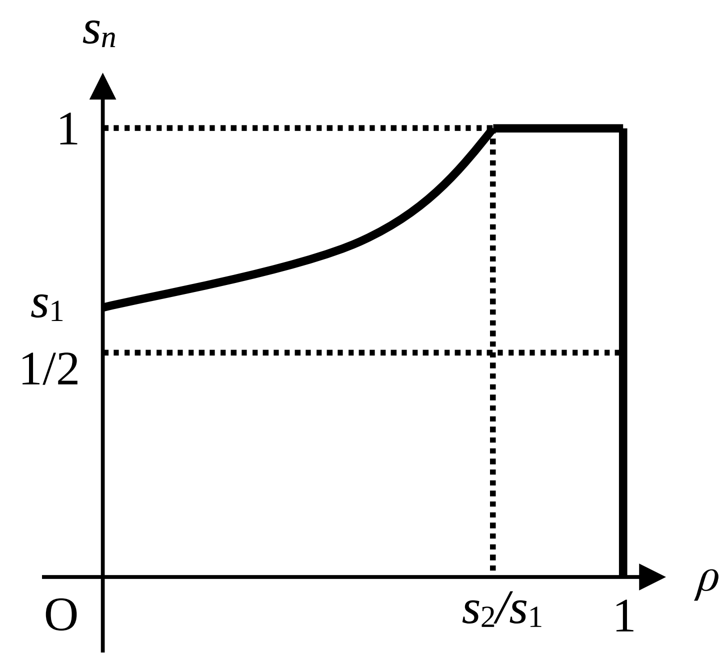
- **In autarky**, the number of firms based in each country is proportional to its size.
- **As trade costs fall,**
  - Horizontally differentiated goods produced in the two countries mutually penetrate each other's home markets (Two-way flows of goods).
  - Firm distribution becomes increasingly skewed toward the larger country *(Home Market Effect and its Magnification)*

Two Parameters: s_1 & ρ

$s_1 = 1 - s_2 \in [1/2, 1)$:
Bigger country's share in market size

$\rho \equiv (\tau)^{1-\sigma} \in [0,1)$: Degree of Globalization:
inversely related to the iceberg cost, $1 < \tau \leq \infty$

s_n: Bigger country's share in firm distribution



Our Main Results: By combining DJ (1992) and HK (1985):

- **2D state space**: (Measures of competitive varieties in the two countries)

- Unique equilibrium path obtained by iterating a **2D-PWS, noninvertible map** with ***four parameters***: θ & δ & s_1 & ρ
  One unit of competitive varieties = θ (> 1) units of monopolistic varieties
  One unit of foreign varieties = ρ (< 1) unit of domestic varieties

- **In autarky** ($\rho = 0$), the dynamics of the two are **decoupled.** Whether they may converge to either synchronized or asynchronized 2-cycles depends on how you draw the initial condition.

- **As trade costs fall** (a higher ρ), they become more **synchronized**:
  - *Basin of attraction* for asynchronized 2-cycles ***shrinks*** and ***disappears***
  - *Basin of attraction* for synchronized 2-cycles ***expands***

Full synchronization is reached with partial trade integration ($\rho < 1$ or $\tau > 0$)

- Fully synchronized at a larger trade cost if country sizes are more unequal
- Even a small size difference spends up synchronization significantly
- The larger country sets the tempo of global innovation cycles, with the smaller country adjusting its rhythm.

2D Dynamical System; $n_{t+1} = F(n_t)$ with $n_t \equiv (n_{1t}, n_{2t}) \in R_+^2$**;**
$(0 < \delta < 1;\ 1 < \theta < e;\ 0 \le \rho < 1;\ 1/2 \le s_1 < 1)$

$$n_{1t+1} = \delta\left(\theta s_1(\rho) + (1-\theta)n_{1t}\right) \qquad \text{if } n_t \in D_{LL} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_j \le s_j(\rho)\right\}$$
$$n_{2t+1} = \delta\left(\theta s_2(\rho) + (1-\theta)n_{2t}\right)$$

$$n_{1t+1} = \delta n_{1t} \qquad \text{if } n_t \in D_{HH} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_j \ge h_j(n_k)\right\}$$
$$n_{2t+1} = \delta n_{2t}$$

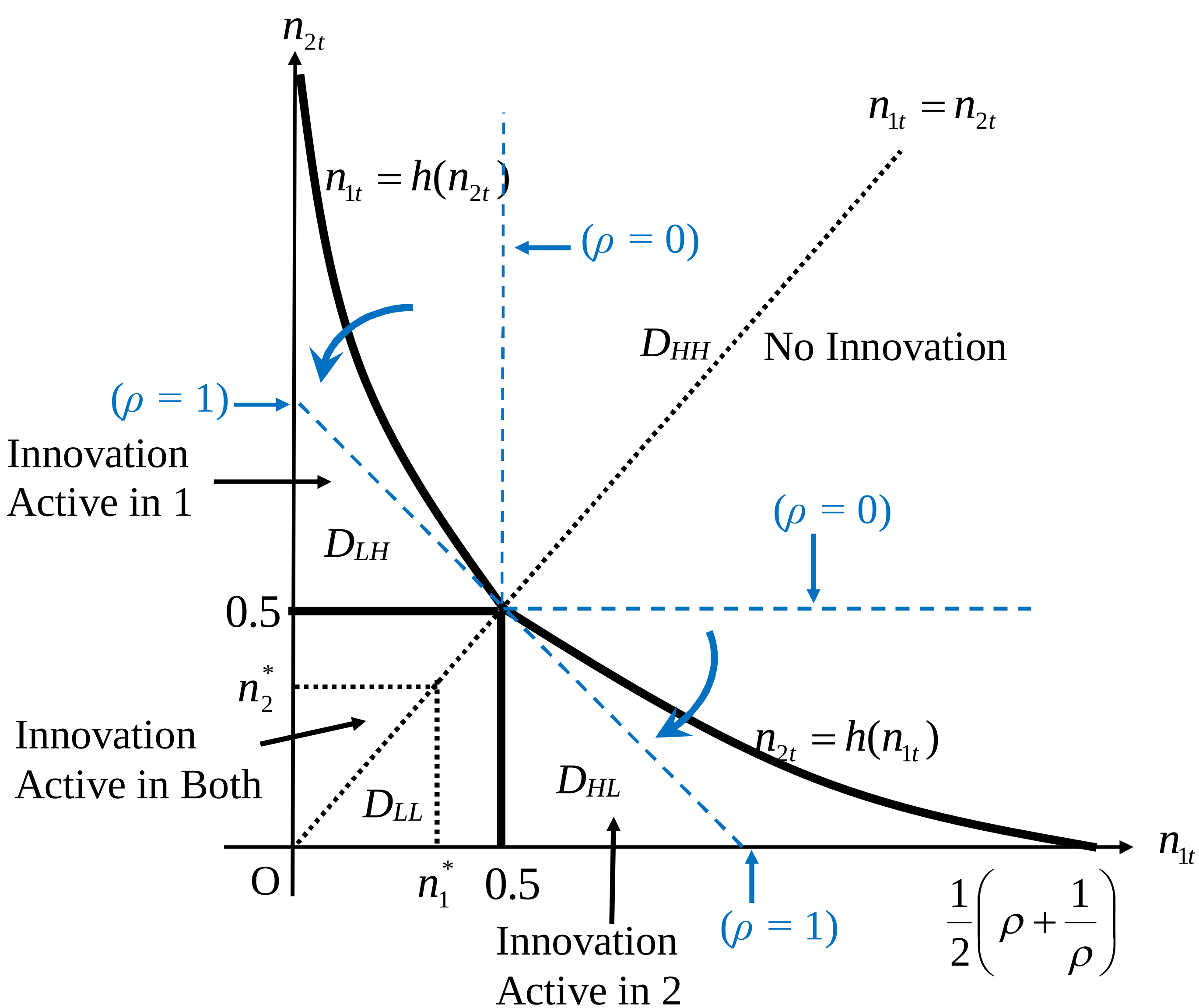
$$n_{1t+1} = \delta n_{1t} \qquad \text{if } n_t \in D_{HL} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 \ge s_1(\rho); n_2 \le h_2(n_1)\right\}$$
$$n_{2t+1} = \delta\left(\theta h_2(n_{1t}) + (1-\theta)n_{2t}\right)$$

$$n_{1t+1} = \delta\left(\theta h_1(n_{2t}) + (1-\theta)n_{1t}\right) \qquad \text{if } n_t \in D_{LH} \equiv \left\{(n_1, n_2) \in R_+^2 \middle| n_1 \le h_1(n_2); n_2 \ge s_2(\rho)\right\}$$
$$n_{2t+1} = \delta n_{2t}$$

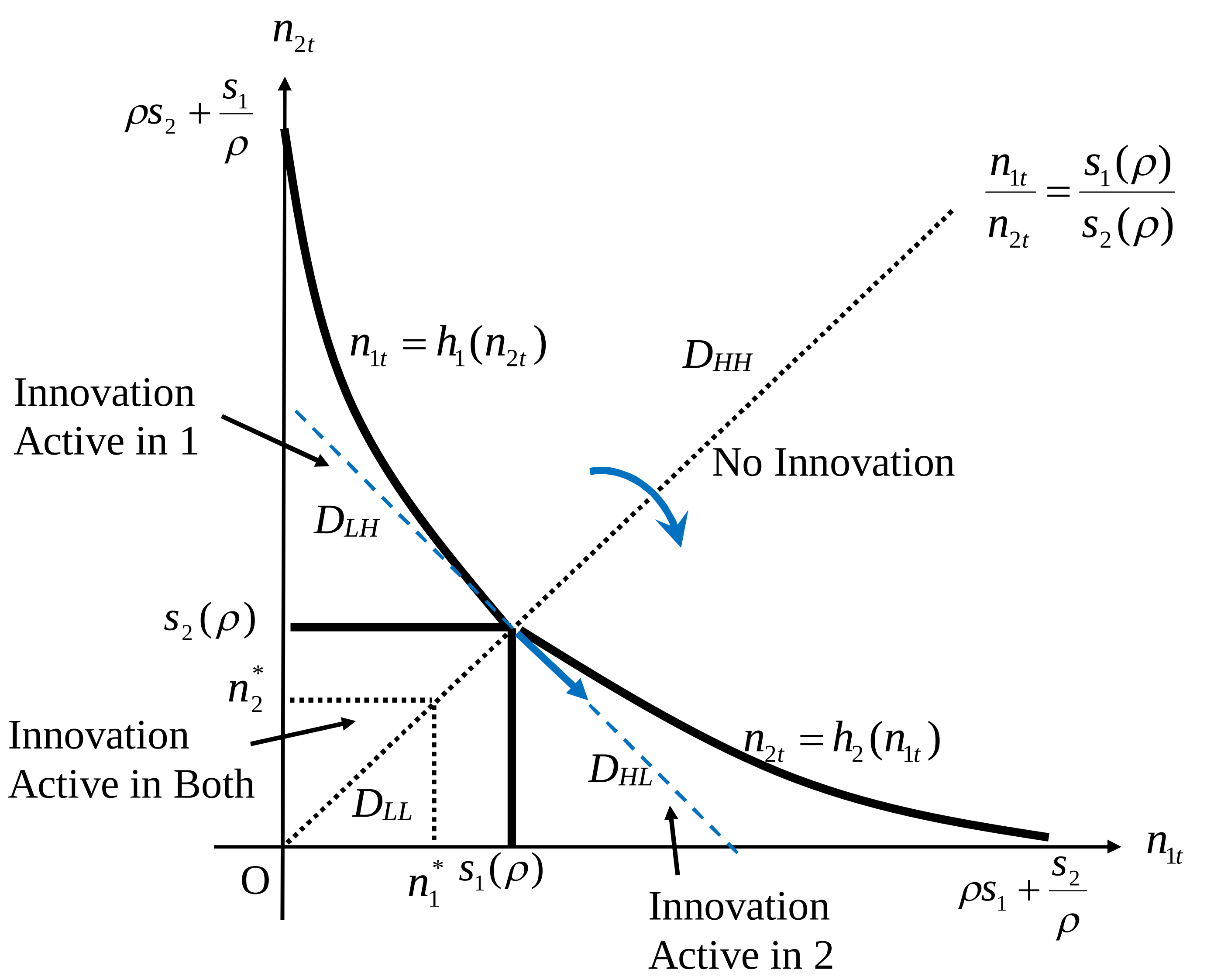
where $s_1(\rho) = 1 - s_2(\rho) = \min\left\{\dfrac{s_1 - \rho s_2}{1-\rho}, 1\right\}$, $\quad 0.5 \le s_1 = 1 - s_2 < 1$;

$h_j(n_k) > 0$ defined implicitly by $\dfrac{s_j}{h_j(n_k) + \rho n_k} + \dfrac{s_k}{h_j(n_k) + n_k/\rho} = 1.$

State Space & Four Domains for the Symmetric Case: $0 < \rho < s_2 / s_1 = 1$



State Space & Four Domains for the Asymmetric Case: $0 < \rho < s_2 / s_1 < 1$



Synchronized vs. Asynchronized 2-Cycles in Autarky: $\rho = 0$; $\delta(\theta-1) > 1 > \delta^2(\theta-1)$,

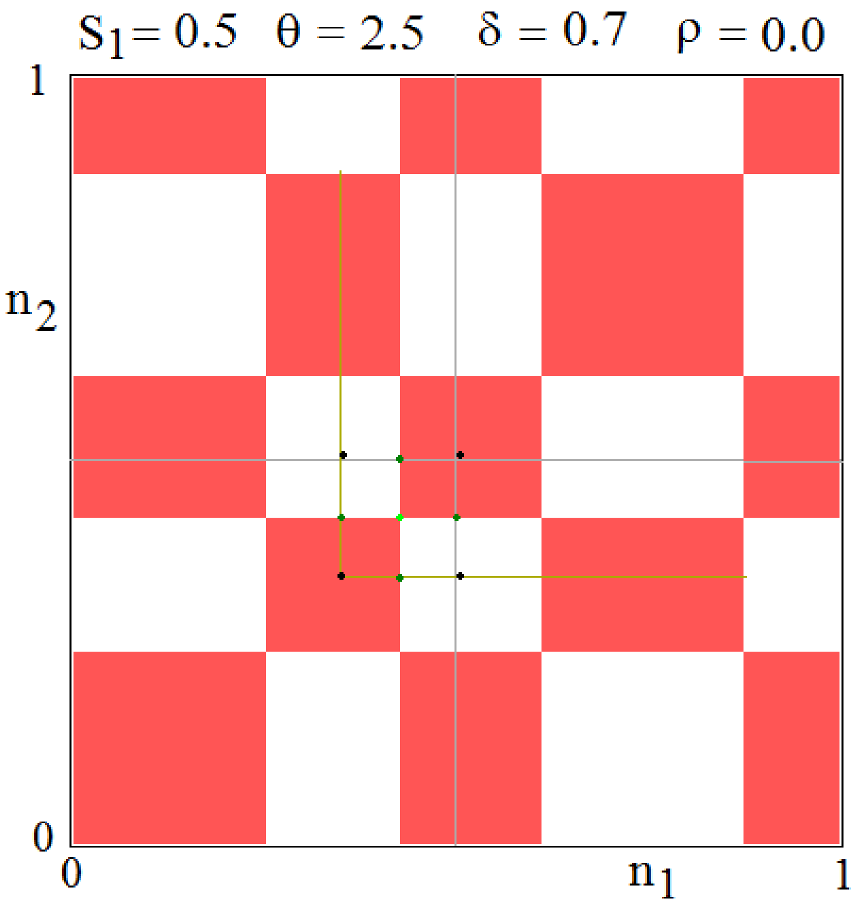
As a 2D-map, this system has

- An unstable steady state; (n_1^*, n_2^*)

- **A pair of stable 2-cycles**
  - **Synchronized;** $(n_{1L}^*, n_{2L}^*) \leftrightarrow (n_{1H}^*, n_{2H}^*)$,
    Basin of Attraction in red.
  - **Asynchronized;** $(n_{1L}^*, n_{2H}^*) \leftrightarrow (n_{1H}^*, n_{2L}^*)$,
    Basin of Attraction in white

- **A pair of saddle 2-cycles:**

$(n_{1L}^*, n_2^*) \leftrightarrow (n_{1H}^*, n_2^*)$; $(n_1^*, n_{2H}^*) \leftrightarrow (n_1^*, n_{2L}^*)$



Symmetric Interdependent 2-Cycles, $s_1 = 0.5$, $\rho \in (0,1)$, $\delta(\theta-1) > 1 > \delta^2(\theta-1)$:

Each component 1D-map has:

- **an *unstable* steady state,** $n_j^* = n^* \equiv \dfrac{\theta\delta/2}{1+(\theta-1)\delta}$ &
- **a *stable* 2-cycle,** $n_{jL}^* = n_L^* \equiv \dfrac{\delta^2\theta/2}{1+(\theta-1)\delta^2} \leftrightarrow n_{jH}^* = n_H^* \equiv \dfrac{\delta\theta/2}{1+(\theta-1)\delta^2}$.

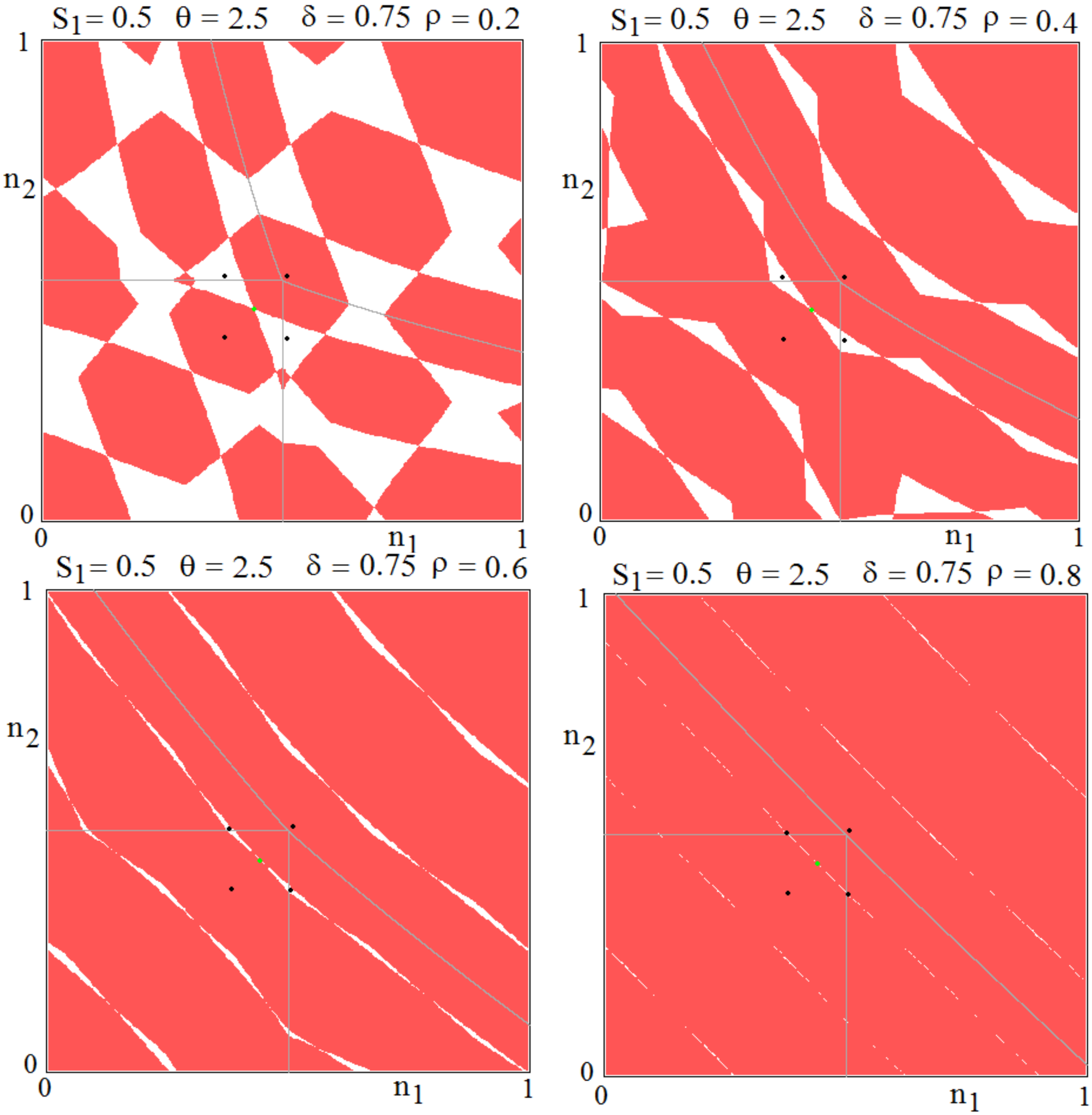
As a 2D-map,

- **Synchronized 2-cycle,** $(n_L^*, n_L^*) \in D_{LL} \leftrightarrow (n_H^*, n_H^*) \in D_{HH}$, is unaffected by $\rho \in (0,1)$.
- **Symmetric Asynchronized 2-cycle,** $(n_L^a, n_H^a) \in D_{LH} \leftrightarrow (n_H^a, n_L^a) \in D_{HL}$, depends on $\rho \in (0,1)$, no longer equal to $(n_L^*, n_H^*) \leftrightarrow (n_H^*, n_L^*)$. It exists for all $\rho \in (0,1)$; stable for $\rho \in (0, \rho_c)$ and **unstable** for $\rho \in (\rho_c, 1)$.

Furthermore, one could see numerically,

- For, $\rho \in (0, \rho_c)$, a higher ρ expands the basin of attraction for the synchronized 2-cycle, and reduces that for the asynchronized 2-cycle.

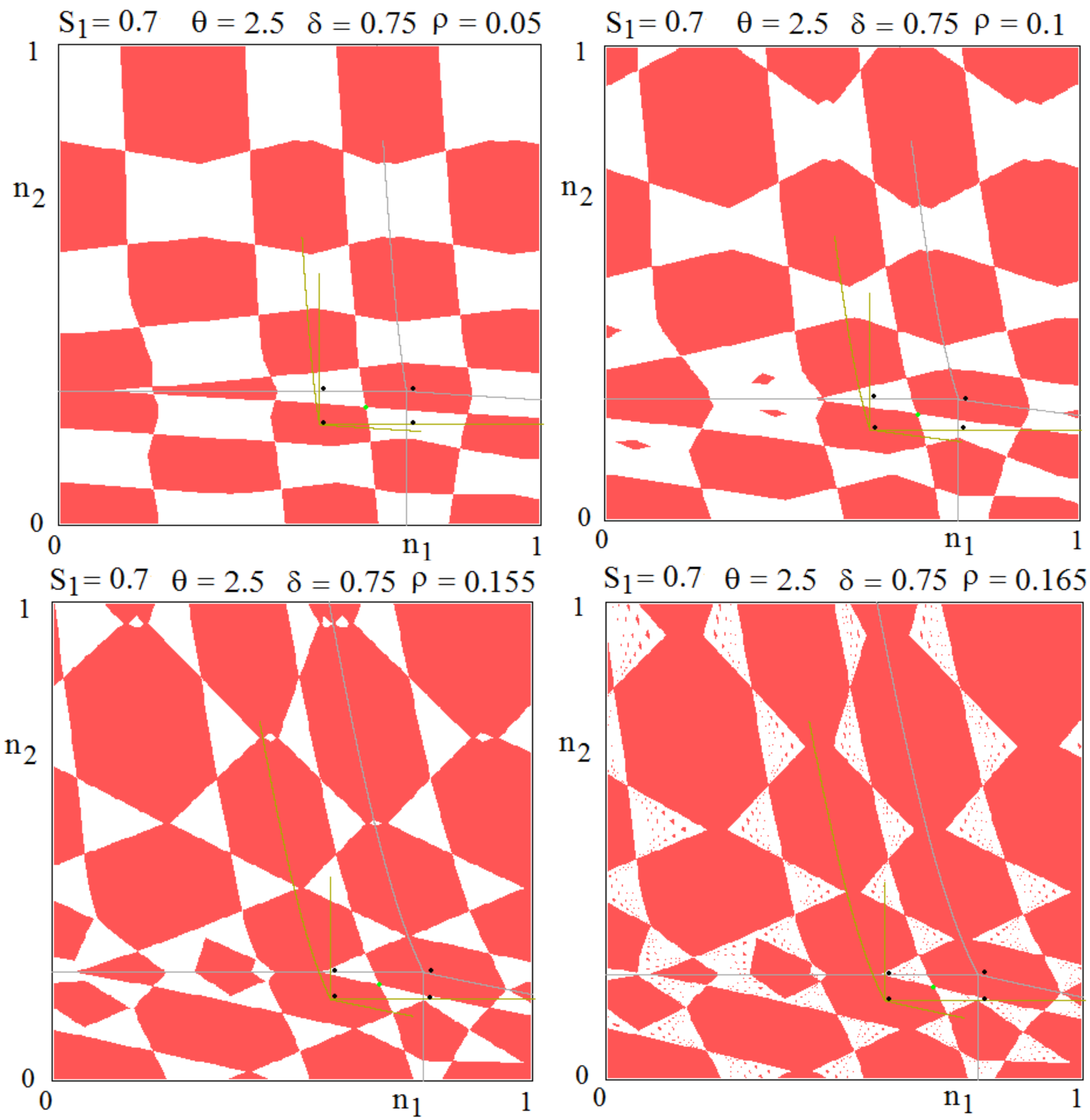
Symmetric Synchronized & Asynchronized 2-Cycles: $s_1 = 0.5;\ \theta = 2.5;\ \delta = 0.75$



Red (Sync. 2-cycle) becomes dominant.

Sym. Async. 2-cycle becomes
a node at ρ = .817867,
a saddle at ρ = .833323.

Asymmetric Synchronized & Asynchronized 2-Cycles $s_1 = 0.7$, $\theta = 2.5$; $\delta = 0.75$



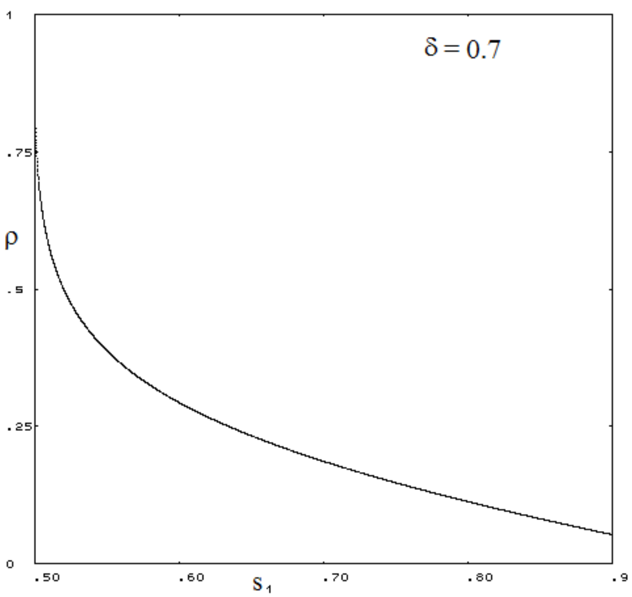
By ρ = .165, infinitely many Red islands appear inside White.

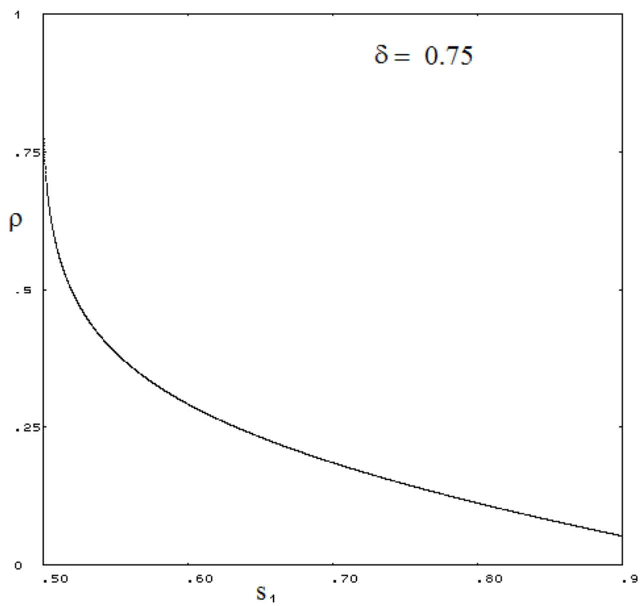
By ρ =.19, the stable asynchronized 2-cycle collides with its basin boundary and disappears, leaving **the Synchronized 2-cycle as the unique attractor**.

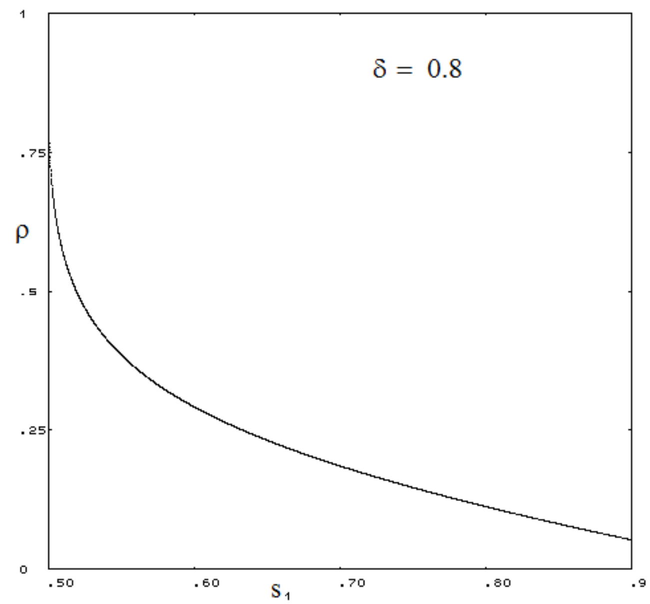
A Smaller Reduction in τ Synchronizes Innovation Cycles with Country Size Asymmetries

Critical Value of ρ_c at which the Stable Asynchronized 2-cycle disappears (as a function of s_1)

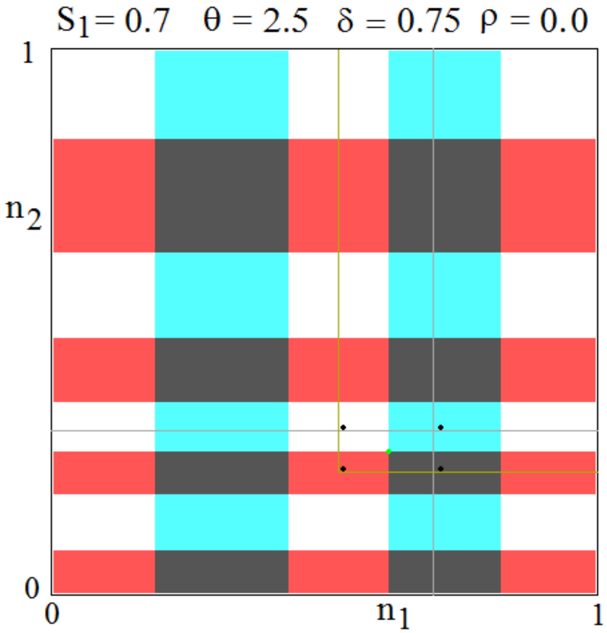
- It declines very rapidly as s_1 increases from 0.5.
- It hardly changes with δ**.**

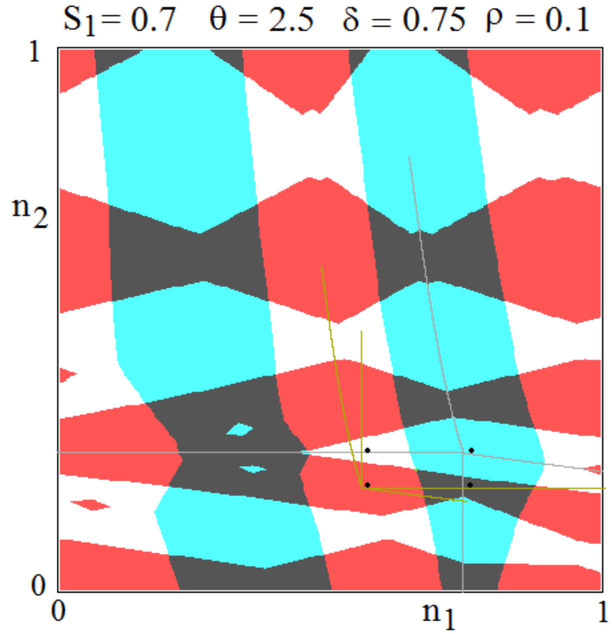


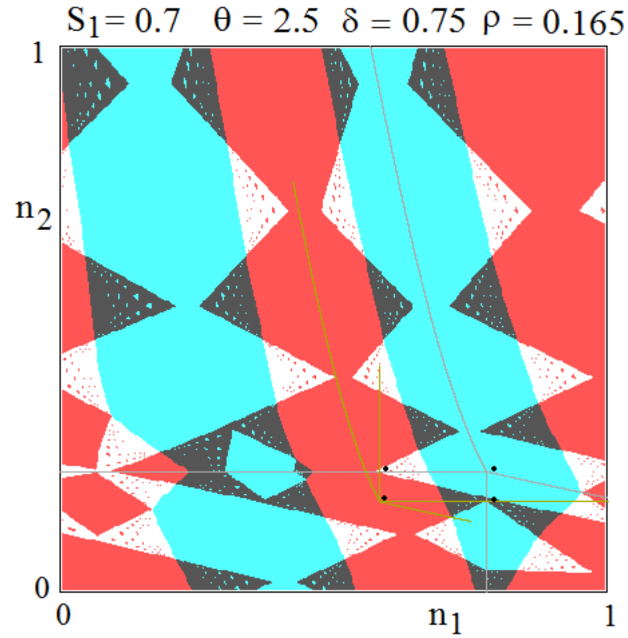


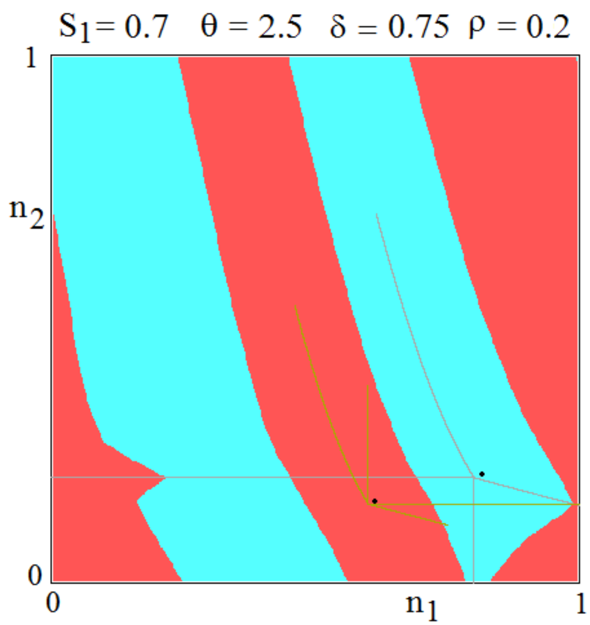


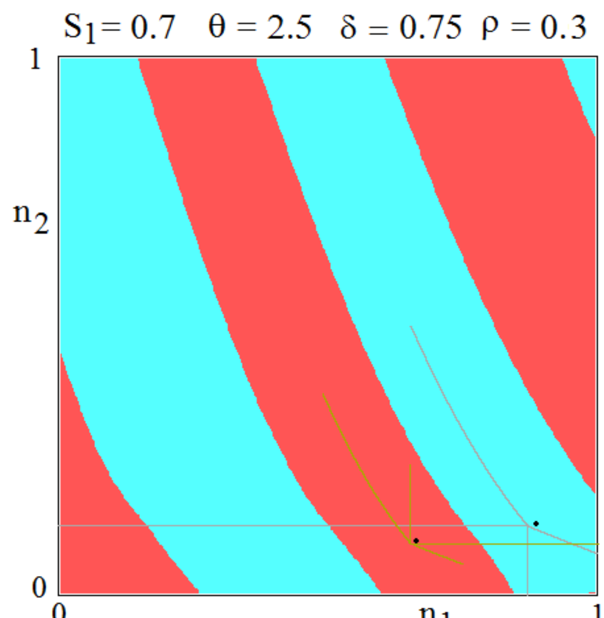
Four Basins of Attraction ($s_1 = 0.7$, $\theta = 2.5$, $\delta = 0.75$)







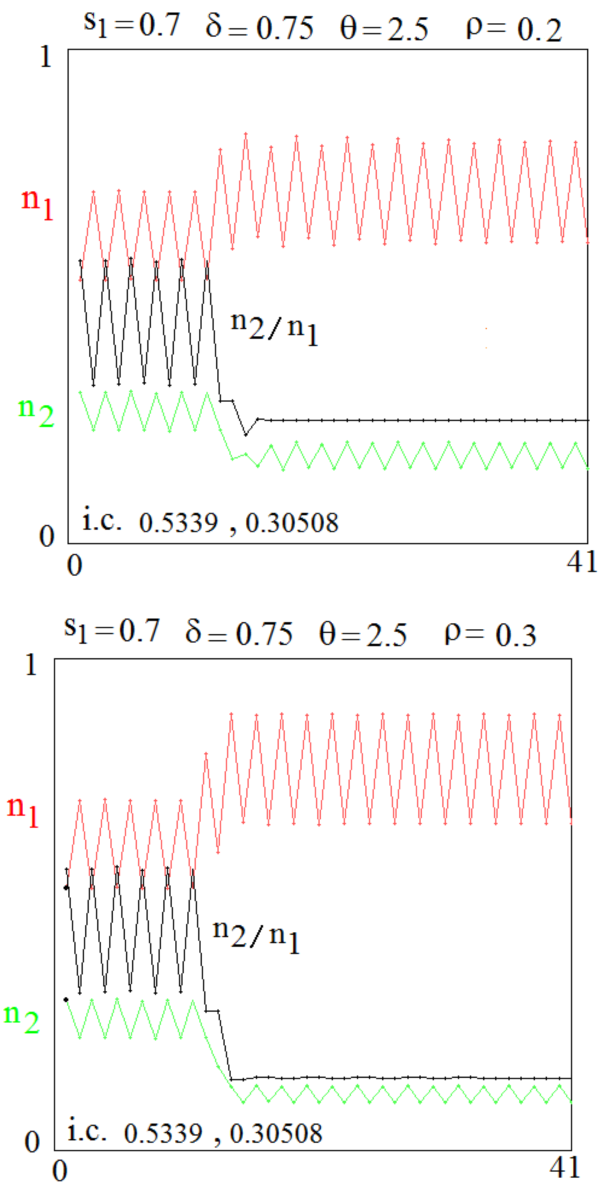




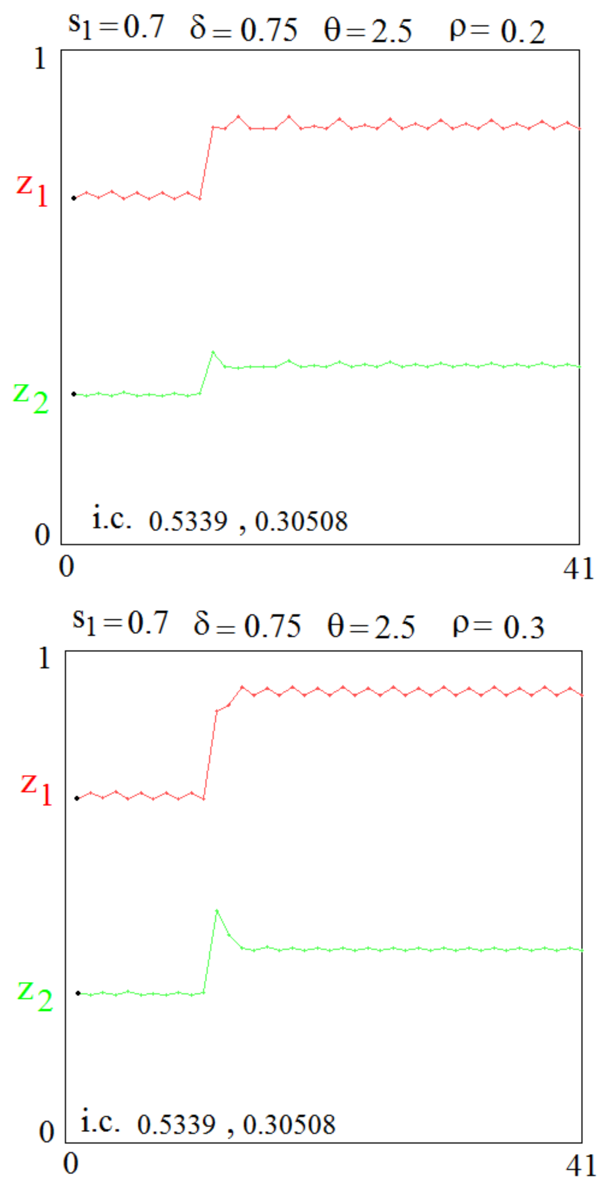
As ρ rises,
Red invades White, and
Azure invades Gray, and
vertical slips of Red and
Azure emerge.

Three Effects of Globalization:

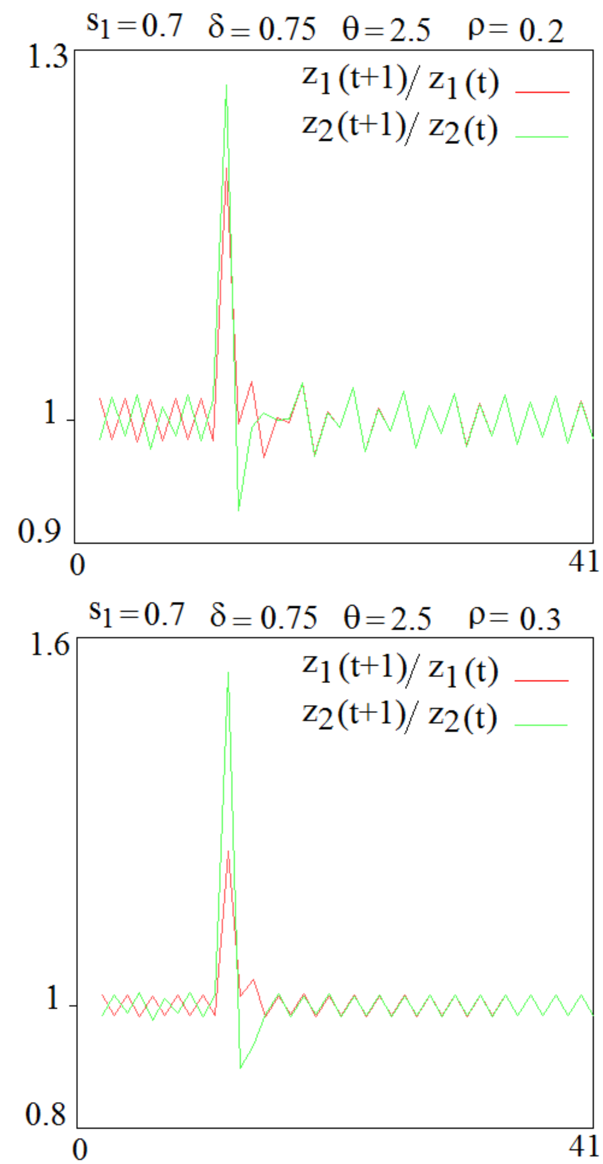
Home Market Effect



Productivity Gains



Synchronization



Summary:

- A 2-sector extension with CES preferences over the two sectors
  - For Cobb-Douglas ($\varepsilon = 1$), innovation dynamics of the two sectors are decoupled.
  - For $\varepsilon > 1$, *synchronized to amplify* fluctuations; for $\varepsilon < 1$, *asynchronized to moderate*
- A 2-country extension with trade between *structurally identical* countries, where the degree of trade globalization ρ acts as a coupling parameter
  - In autarky ($\rho = 0$), innovation dynamics of the two countries are decoupled.
  - As trade cost falls, they become more synchronized
  - Full synchronization occurs at a strictly positive trade cost (and at a larger trade cost with more unequal country sizes)
  - The smaller country adjusts its rhythm to the rhythm of the bigger country.

More to Come:

- Synchronization of ***chaotic fluctuations***
- ***More sectors*** *or* ***more countries***
- A ***2-sector & 2-country extension*** to study the effects of globalization between two *structurally dissimilar* countries
  - Two Industries: **Upstream** & **Downstream,** each produces DS composite as in DJ.
  - One country has comparative advantage in **U**; the other in **D**
  - *My conjecture*: Globalization leads to an asynchronization

  Consistent with the empirical evidence (Trade causes synchronization among developed countries, but *not* between developed and developing countries)